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Local Nikolskii Constants for a Special Class of Baskakov Operators

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1. DEFINITIONS AND AUXILIARY RESULTS

DEFINITION 1.1. Let $(\Phi_n)_{n\in\mathbb{N}}$, Φ_n: $[0, b] \to \mathbb{R}$ $(b > 0)$ be a sequence of functions, having the following properties:

(i) Φ_n is infinitely differentiable on $[0, b]$;

(ii) $\Phi_n(0) = 1$;

(iii) Φ_n is completely monotone on $[0, b]$, i.e., $(-1)^k \Phi_n^{(k)}(x) \geqslant 0$ for $x \in [0, b]$ and $k \in \mathbb{N}_0$;

(iv) there exists an integer c, such that

$$\Phi_n^{(k)}(x) = -n\Phi_{n-c}^{(k-1)}(x)$$

for $x \in [0, b]$, $k \in \mathbb{N}$, $n \in \mathbb{N}$, $n > \max(c, 0)$.

Then the sequence $(\Phi_n)_{n\in\mathbb{N}}$ generates two sequences of operators, namely,

$$T_n(f; x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Phi_n^{(k)}(x)\, x^k f\left(\frac{k}{n}\right), \qquad x \in [0, b],\ n \in \mathbb{N}, \quad (1.1)$$

and

$$\tilde{T}_n(g; x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Phi_n^{(k)}(x) x^k \frac{1}{2} \left\{ g\left(\frac{k}{n}\right) + g\left(2x - \frac{k}{n}\right) \right\}, \qquad x \in [0, b],\ n \in \mathbb{N}. \quad (1.2)$$

Remarks. (1) It can be shown that operators (1.1) specialize some well-known operators as those of Baskakov [1] and Schurer [9].

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(2) A remark of Schurer [9; p. 23] says that the function can be continued analytically to a function Θ_n, which is holomorphic in the closed disk $B := \{z \in \mathbb{C}: |z-b| \leqslant b\}$. A complete proof of this fact is given in [2; p. 47 ff.].

Let now $X \subset \mathbb{R}$ be an interval, then $C_M(X)$ denotes the space of all functions $f \in C(X)$ such that

$$|f(t)| \leqslant A(f) + B(f)\,|t|^{m(f)}$$

for some constants $A(f), B(f) \in \mathbb{R}^+$ and $m(f) \in \mathbb{N}_0$.

THEOREM 1.2. (a) $(T_n)_{n\in\mathbb{N}}$ *is a sequence of linear positive operators from* $C_M[0, \infty)$ *in* $C[0, b]$ *with the property*

$$\lim_{n\to\infty} T_n(f; x) = f(x), \qquad f \in C_M[0, \infty),\ x \in [0, b].$$

(b) $(\tilde{T}_n)_{n\in\mathbb{N}}$ *is a sequence of linear positive operators from* $C_M(\mathbb{R})$ *in* $C[0, b]$ *with the property*

$$\lim_{n\to\infty} \tilde{T}_n(g; x) = g(x), \qquad g \in C_M(\mathbb{R}),\ x \in [0, b].$$

Proof. Part (a) follows immediately from a theorem of Rathore (cf. [6; pp. 35–39]). Moreover (b) follows from (a) and the fact that

$$\tilde{T}_n(g; x) = \tfrac{1}{2}\{T_n(g; x) + T_n(g(2x-t); x)\}, \qquad x \in [0, b]. \quad \blacksquare$$

For $n \in \mathbb{N}$, $s \in \mathbb{N}_0$ and $x \in [0, b]$ we write

$$T_n((t-x)^s; x) = \frac{1}{n^s} M_{n,s}(x), \tag{1.3}$$

where

$$M_{n,s}(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Phi_n^{(k)}(x)\, x^k (k-nx)^s. \tag{1.4}$$

Then the following result of Sikkema [10; Satz 4; p. 236] is valid.

LEMMA 1.3. *For* $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $n > \max(c, 0)$ *and* $x \in [0, b]$ *we have*

$$M_{n,m+1}(x) = nx \sum_{s=0}^{m} \binom{m}{s} (1-cx)^{m-s} M_{n-c,s}(x) - nx M_{n,m}(x). \tag{1.5}$$

From the recurrence relation (1.5) we obtain

LEMMA 1.4. *For* $m \in \mathbb{N}$, $n \in \mathbb{N}$, $n > \max(c, 0)$ *and* $x \in [0, b]$ *the formula*

$$M_{n,m}(x) = \sum_{j=0}^{[m/2]} \Psi_{m,j}(x) n^j \tag{1.6}$$

holds, where $\Psi_{m,j}$ $(0 \leqslant j \leqslant [m/2])$ *is an algebraic polynomial of degree* m *in* x. *Moreover there exists a positive constant* $C(m, b)$ *such that*

$$|M_{n,m}(x)| \leqslant C(m, b) n^{[m/2]}$$

and

$$|T_n((t-x)^m; x)| \leqslant C(m, b)\, n^{-[(m+1)/2]}$$

hold uniformly for all $x \in [0, b]$ *and* $n > \max(mc, 0)$.

Proof. To prove Lemma 1.4 we have only to show that formula (1.6) is valid. This will be done by means of mathematical induction. According to Baskakov [1; p. 249 f.] we have

$$M_{n,0}(x) = T_n(1; x) = 1.$$

Now let us assume that for $0 \leqslant r < m$

$$M_{n,r}(x) = \sum_{j=0}^{[r/2]} \Psi_{r,j}(x) n^j,$$

where $\Psi_{r,j}$ $(0 \leqslant j \leqslant [r/2])$ is an algebraic polynomial of degree r in x.

By Lemma 1.3 we obtain

$$\begin{aligned} M_{n,m}(x) &= nx \sum_{s=0}^{m-1} \binom{m-1}{s} (1-cx)^{m-1-s} M_{n-c,s}(x) - nxM_{n,m-1}(x) \\ &= nx \sum_{s=0}^{m-2} \binom{m-1}{s} (1-cx)^{m-1-s} M_{n-c,s}(x) \\ &\quad + nx[M_{n-c,m-1}(x) - M_{n,m-1}(x)] \\ &= \sum_{s=0}^{m-2} \sum_{j=0}^{[s/2]} \binom{m-1}{s} \Psi_{s,j}(x)(1-cx)^{m-1-s} nx(n-c)^j \\ &\quad + \sum_{j=0}^{[(m-1)/2]} \sum_{k=0}^{j-1} \binom{j}{k} (-c)^{j-k} \Psi_{m-1,j}(x) n^k. \end{aligned} \tag{1.7}$$

We can see from (1.7) that the degree in n of $M_{n,m}(x)$ is equal to $[m/2]$. Moreover we know that $\Psi_{s,j}$ is an algebraic polynomial of degree s in x for all $0 \leqslant j \leqslant [s/2]$ $(0 \leqslant s \leqslant m-1)$. Thus we can write

$$M_{n,m}(x) = \sum_{j=0}^{[m/2]} \Psi_{m,j}(x) n^j$$

with suitable polynomials $\Psi_{m,j}$ of degree m in x. ∎

THEOREM 1.5 (Conclusions from Definition 1.1).

(a) *The case* $c \in \mathbb{Z}$, $c > 0$. (i) *Suppose, there is a sequence* $(\Phi_n)_{n \in \mathbb{N}}$, *which satisfies the conditions of Definition* 1.1 *on an interval* $[0, b]$ *with* $c > 0$, *then it must be* $0 \leqslant b \leqslant 1/c$.

(ii) *For each sequence* $(\Phi_n)_{n \in \mathbb{N}}$, *which satisfies the conditions of Definition* 1.1 *on an interval* $[0, b]$ *with* $c > 0$, *we have*

$$\Phi_{cm+j}(x) = (1-cx)^{m+j/c} + \frac{(-1)^m}{(m-1)!} \prod_{k=1}^{m} (ck+j) \int_0^x (x-t)^{m-1} \{\Phi_j(t) - (1-ct)^{j/c}\}\, dt,$$

$$j = 1, 2, \ldots, c;\ m \in \mathbb{N};\ x \in [0, b]. \quad (1.8)$$

(iii) *The sequence* $\Phi_n(x) = (1-x)^n$ *satisfies all conditions of Definition* 1.1 *with* $c = 1$ *on the interval* $[0, 1]$. *However, for* $c > 0$ *the sequence* $\Phi_n(x) = (1-cx)^{n/c}$ *does not satisfy all conditions of Definition* 1.1 (*cf. a remark of Schurer in* [9; p. 66]).

(b) *The case* $c = 0$. *In case* $c = 0$ *the conditions of Definition* 1.1 *are only satisfied for the sequence* $\Phi_n(x) = \exp(-nx)$, *namely, on any interval* $[0, b]$ *with* $b > 0$. *Hence the corresponding operators* T_n *and* $\tilde{T}_n$ *are defined for all* $x \geqslant 0$.

(c) *The case* $c \in \mathbb{Z}$, $c < 0$. *Suppose, there is a sequence* $(\Phi_n)_{n \in \mathbb{N}}$, *which satisfies the conditions of Definition* 1.1 *on an interval* $[0, b]$ *with* $c < 0$, *then we have* $\Phi_n(x) = (1-cx)^{n/c}$. *On the other hand, the sequence* $\Phi_n(x) = (1-cx)^{n/c}$ *satisfies the conditions of Definition* 1.1 *on any interval* $[0, b]$ *with* $b > 0$. *Thus the corresponding operators* T_n *and* $\tilde{T}_n$ *are defined for all* $x \geqslant 0$.

Proof. (a) (i) From the positivity of the operator T_n we get

$$T_n((t-b)^2; b) = \frac{b(1-cb)}{n} \geqslant 0,$$

i.e., $b(1-cb) \geqslant 0$ and thus $0 \leqslant b \leqslant 1/c$.

(ii) Formula (1.8) follows from property (iv) of Definition 1.1, cf. Schurer [9; p. 63 f.].

(b) The initial value problem $\Phi_n(x) = -n\Phi_n(x)$, $\Phi_n(0) = 1$ has a unique solution, which is given by $\Phi_n(x) = \exp(-nx)$.

(c) As we have seen, the function Φ_n can be continued analytically to a function Θ_n, which is holomorphic on $|z - b| \leqslant b$. Thus, expanding Θ_n in a Taylor series, we have for a suitable $R > 0$

$$\Theta_n(z) = \sum_{k=0}^{\infty} \frac{\Phi_n^{(k)}(0)}{k!} z^k, \qquad |z| < R.$$

Moreover we obtain from property (iv) of Definition 1.1

$$\Phi_n^{(k)}(x) = -n\Phi_{n-c}^{(k-1)}(x) = \cdots = (-1)^k \prod_{i=0}^{k-1} (n - ic)\, \Phi_{n-kc}(x),$$

and hence

$$\Phi_n^{(k)}(0) = (-1)^k \prod_{i=0}^{k-1} (n - ic).$$

Therefore

$$\begin{aligned}\Theta_n(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k n(n-c) \cdots (n-(k-1)c)}{k!} z^k \\ &= (1 - cz)^{n/c}, \qquad |z| < R\end{aligned}$$

and thus by means of analytic continuation

$$\Theta_n(z) = (1 - cz)^{n/c} \qquad \text{for} \quad |z - b| \leqslant b. \quad \blacksquare$$

2. An Asymptotic Estimation for $T_n(|t - x|^\beta; x)$

In this section we will show that the asymptotic estimation

$$T_n(|t - x|^\beta; x) \cong \frac{\Gamma((\beta + 1)/2)}{\sqrt{\pi}} \left[\frac{2x(1 - cx)}{n}\right]^{\beta/2}, \qquad n \to \infty \tag{2.1}$$

holds for all $\beta > 0$ and all x in a suitable interval.

Now, using the result of [3; Theorem 3.2] and Lemma 1.4, we obtain

THEOREM 2.1. *Let* $(\Phi_n)_{n\in\mathbb{N}}$ *be a sequence of functions, which satisfies the conditions of Definition* 1.1 *on an interval* $[0, b]$. *Suppose, for a real* α, $\alpha < \frac{1}{2}$ *and all* $x \in [0, b]$ *the asymptotic relation*

$$\frac{(-1)^k}{k!}\,\Phi_n^{(k)}(x)\,x^k \cong \frac{1}{\sqrt{n}}\,\frac{1}{\sqrt{2\pi x(1-cx)}}\exp\left[\frac{-n}{2x(1-cx)}\left(\frac{k}{n}-x\right)^2\right],$$

$n \to \infty$, *holds uniformly for all* $k \in A_n(x) := \{k \in \mathbb{Z}\colon |k/n - x| < n^{-\alpha}\}$, *then for all* $\beta > 0$ *and all* $x \in [0, b]$ *the estimation*

$$T_n(|t-x|^\beta; x) \cong \frac{\Gamma((\beta+1)/2)}{\sqrt{\pi}}\left[\frac{2x(1-cx)}{n}\right]^{\beta/2}, \qquad n \to \infty$$

is valid.

Remark. For the Bernstein- and Szász–Mirakjan-operators the result of Theorem 2.1 was already probed by Rathore in [5].

THEOREM 2.2. (a) *Let* $(\Phi_n)_{n\in\mathbb{N}}$ *be a sequence of functions, which satisfies the conditions of Definition* 1.1 *on an interval* $[0, b]$ *with* $c > 0$, *then estimation* (2.1) *holds for all* $0 < x < \min(b, 1/(2c))$ *and all* $\beta > 0$.

(b) *In case of Bernstein-operators (i.e.,* $\Phi_n(x) = (1-cx)^n$) (2.1) *is valid for all* $\beta > 0$ *and all* $0 < x < 1$.

(c) *We have learned from Theorem* 1.5 *that in case* $c \leqslant 0$ *the conditions of Definition* 1.1 *are only satisfied by the sequences* $\Phi_n(x) = \exp(-nx)$ $(c = 0)$ *and* $\Phi_n(x) = (1-cx)^{n/c}$ $(c < 0)$, *namely, on any interval* $[0, b]$ $(b > 0)$. *Then* (2.1) *holds for all* $x > 0$ *and all* $\beta > 0$.

Proof. (a) *Step* I. For $0 < x < \min(b, 1/(2c))$ the asymptotic relation

$$\frac{(-1)^k}{k!}\,\Phi_n^{(k)}(x)x^k \cong \frac{n(n-c)\cdots(n-(k-1)c)}{k!}\,(1-cx)^{n/c-k}x^k, \quad (2.2)$$

$n \to \infty$, holds uniformly for all $k \in A_n(x) := \{k \in \mathbb{Z}\colon |k/n - x| < n^{-\alpha}\}$, where $\frac{1}{3} < \alpha < \frac{1}{2}$.

Proof of Step I. Suppose, we have $n \geqslant n_0(x) \geqslant [2c/(1-cx)]^{1/\alpha}$. Hence, putting $n = cm + j$ $(m \in \mathbb{N}; j = 1, \ldots, c)$, we get

$$\left|\frac{k}{n} - x\right| < n^{-\alpha} \Rightarrow k < \frac{n}{2c} \leqslant \frac{m+1}{2} \leqslant m - 1.$$

Now for $0 \leqslant k \leqslant m-1$ formula (1.8) of Theorem 1.5 yields

$$\frac{(-1)^k}{k!}\Phi^{(k)}_{cm+j}(x)x^k$$
$$= \frac{(j+cm)(j+c(m-1))\cdots(j+c(m-k+1))}{k!}(1-cx)^{(j+c(m-k))/k}x^k$$
$$+\frac{(-1)^{m+k}}{(m-1-k)!\,k!}\prod_{i=1}^{m}(ci+j)x^k\int_0^x(x-t)^{m-1-k}\{\Phi_j(t)-(1-ct)^{j/c}\}\,dt. \tag{2.3}$$

Moreover

$$\Gamma_m(x) := \frac{\prod_{i=1}^{m-k}(ci+j)}{(m-1-k)!\,(1-cx)^{m-k+j/c}} \times \int_0^x (x-t)^{m-1-k}\underbrace{|\Phi_j(t)-(1-ct)^{j/c}|}_{\leqslant D}\,dt$$
$$\leqslant D\frac{\prod_{i=1}^{m-k}(ci+j)x^{m-k}}{(m-1-k)!\,(1-cx)^{m-k+j/c}}$$
$$\leqslant D(m-k+j/c)\frac{(cx)^{m-k}}{(1-cx)^{m-k+j/c}}$$
$$= D(m-k+j/c)(cx)^{-j/c}\left[\frac{cx}{1-cx}\right]^{(cm+j)/c-k}$$
$$\leqslant D(cx)^{-j/c}(m+j/c)\left[\frac{cx}{1-cx}\right]^{(cm+j)/c-k}$$
$$\leqslant D(cx)^{-j/c}(m+j/c)\left[\frac{cx}{1-cx}\right]^{((cm+j)/c)(1-2cx)} \to 0, \quad m\to\infty, \tag{2.4}$$

uniformly for all $k \in A_n(x)$, because

$$0 < x < \min(b, 1/(2c)) \Rightarrow 0 < \frac{cx}{1-cx} < 1.$$

From (2.3) and (2.4) we receive (2.2). ●

Step II. For $0 < x < \min(b, 1/(2c))$ the asymptotic relation

$$\frac{(-1)^k}{k!}\Phi^{(k)}_n(x)x^k \cong \frac{1}{\sqrt{2\pi x(1-cx)n}}\exp\left[\frac{-n}{2x(1-cx)}\left(\frac{k}{n}-x\right)^2\right], \quad n\to\infty, \tag{2.5}$$

holds uniformly for all $k \in A_n(x)$, where $\frac{1}{3} < \alpha < \frac{1}{2}$.

Proof of Step II. Using Stirling formula we obtain from (2.2)

$$\frac{(-1)^k}{k!}\Phi_n^{(k)}(x)x^k \cong \frac{n(n-c)\cdots(n-(k-1)c)}{k!}(1-cx)^{n/c-k}x^k$$

$$= \frac{\Gamma(n/c+1)}{\Gamma(n/c-(k-1))k!}(1-cx)^{n/c-k}x^k$$

$$\cong \frac{(n/c+1)^{n/c+1/2}(1-cx)^{n/c-k}x^k}{k^{k+1/2}(n/c-(k-1))^{n/c-k+1/2}\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n+c}{k(n-kc+c)}}\left[\frac{(n+c)(1-cx)}{n+c-kc}\right]^{(n-kc)/c}\left[\frac{x(n+c)}{k}\right]^k$$

$$\cong \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{x(1-cx)}}\underbrace{\left[\frac{n(1-cx)}{n-kc}\right]^{(n-kc)/c}\left[\frac{nx}{k}\right]^k}_{=:W_n(x)}, \quad n\to\infty, \tag{2.6}$$

uniformly for $k \in A_n(x)$.

Now

$$-\log W_n(x) = (n/c-k)\log\left[\frac{n-kc}{n(1-cx)}\right] + k\log\left[\frac{k}{nx}\right]$$

$$= \frac{n}{c}(1-cx)\left[1-\frac{c}{1-cx}\left(\frac{k}{n}-x\right)\right]\log\left[1-\frac{c}{1-cx}\left(\frac{k}{n}-x\right)\right]$$

$$+ nx\left[1+\frac{1}{x}\left(\frac{k}{n}-x\right)\right]\log\left[1+\frac{1}{x}\left(\frac{k}{n}-x\right)\right]. \tag{2.7}$$

By Taylor formula for $|u|<1$

$$\log(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}\frac{u^3}{(1+\Theta u)^3} = u - \frac{1}{2}u^2(1+\varepsilon u),$$

$0<\Theta<1$, where ε remains bounded as $u\to 0$. Substituting in (2.7)

$$-\log W_n(x) = \frac{n}{c}(1-cx)\left[1-\frac{c}{1-cx}\left(\frac{k}{n}-x\right)\right]$$

$$\left[-\frac{c}{1-cx}\left(\frac{k}{n}-x\right)-\frac{c^2}{2(1-cx)^2}\left(\frac{k}{n}-x\right)^2-\frac{\varepsilon}{2}\frac{c^3}{(1-cx)^3}\left(\frac{k}{n}-x\right)^3\right]$$

$$+ nx\left[1+\frac{1}{x}\left(\frac{k}{n}-x\right)\right.$$

$$\left.\times\left[\frac{1}{x}\left(\frac{k}{n}-x\right)-\frac{1}{2x^2}\left(\frac{k}{n}-x\right)^2-\frac{\varepsilon}{2x^3}\left(\frac{k}{n}-x\right)^3\right]\right.$$

$$= \frac{n}{2x(1-cx)}\left(\frac{k}{n}-x\right)^2 + O_x(n^{1-3\alpha}), \qquad n\to\infty,$$

uniformly for $k \in A_n(x)$. Therefore we have uniformly for all $k \in A_n(x)$

$$W_n(x) = \exp\left[\frac{-n}{2x(1-cx)}\left(\frac{k}{n}-x\right)^2\right], \qquad n \to \infty. \tag{2.8}$$

Relation (2.5) now follows by (2.6) and (2.8). ●

An application of Theorem 2.1 completes the proof.

(b) Part (b) of Theorem 2.2 was already proved by Rathore [5: p. 53 f.] (cf. Lorentz [4; pp. 15–17]).

(c) For the sequence $\Phi_n(x) = \exp(-nx)$ the result of part (c) was proved by Rathore [5; p. 40 f.].

For the sequence $\Phi_n(x) = (1-cx)^{n/c}$ $(c < 0)$ the result of part (c) can be proved in the same way as Step II in the proof of part (a). ∎

Remark. Moreover, by means of Lemma 1.4 we can prove the following completion of Theorem 2.2:

Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of functions, which satisfies the conditions of Definition 1.1 on an interval $[0, b]$ with $c \in \mathbb{Z}$, then the asymptotic extension

$$T_n((t-x)^{2m}; x) \cong \frac{1\cdot 3\cdot 5\cdots(2m-1)(x(1-cx))}{n^m}$$

$$= \frac{\Gamma((2m+1)/2)}{\sqrt{\pi}}\left[\frac{2x(1-cx)}{n}\right]^m$$

holds for all $x \in [0, b]$ and all $m \in \mathbb{N}$

3. The Local Nikolskii Constants

Let X, $\hat{X}$ be two subsets of $\mathbb{R}$ with $\hat{X} \subset X$. For a sequence $(L_n)_{n\in\mathbb{N}}$ of operators, defined on the domain $C_M(X)$ into the domain $C(\hat{X})$, the error in approximating a subclass $A \subset C_M(X)$ by L_n, at a given point x is defined to be

$$\Delta(L_n; A; x) := \sup_{f\in A} |L_n(f; x) - f(x)|. \tag{3.1}$$

If there exists a numerical sequence $\Psi_n(L; A) \to 0$ as $n \to \infty$, such that

$$\Delta(L_n; A; x) = C(L; A; x)\,\Psi_n(L; A) + o_x(\Psi_n(L; A)), \qquad n \to \infty, \tag{3.2}$$

where $C(L; A; x)$ is a positive number, then $C(L; A; x)$ is called the local

Nikolskii constant corresponding to the order $\Psi_n(L; A)$ of approximation of the class A, at the point x, by the operator L_n.

In this section A is always one of the following three classes

$$\mathrm{Lip}(C(X); \alpha; 1) := \{f \in C(X): |f(x+h) - f(x)| \leqslant |h|^\alpha, \\ \forall x, x+h \in X\}, \qquad 0 < \alpha \leqslant 1;$$

$$\mathrm{Lip}^*(C(X); \alpha; 2) := \{f \in C(X): |f(x+h) - 2f(x) + f(x-h)| \\ \leqslant 2\,|h|^\alpha, \forall x, x \pm h \in X\}, \qquad 0 < \alpha \leqslant 2;$$

$$W^{(1)}(C(X); \alpha; 1) := \{f \in C^1(X): f' \in \mathrm{Lip}(C(X); \alpha; 1)\}, \qquad 0 < \alpha \leqslant 1.$$

Moreover we want to study the approximation of higher order. Thus we consider the quantity

$$\Delta(L_n; \alpha; q; x) := \sup_{f \in W^{(q)}(C(X); \alpha; 1)} \left| L_n\Big(f(t) - \sum_{k=0}^{q} \frac{f^{(k)}(x)}{k!}(t-x)^k; x\Big)\right|$$

with

$$W^{(q)}(C(X); \alpha; 1) := \{f \in C^{(q)}(X): f^{(q)} \in \mathrm{Lip}(C(X); \alpha; 1)\}, \\ 0 < \alpha \leqslant 1, q \in \mathbb{N}.$$

Now, using estimation (2.1) and the results of [3; Theorems 2.3–2.4], we get (a) for the sequence $(T_n)_{n \in \mathbb{N}}$

	Order of approximation	Local Nikolskii constant
$\Delta(T_n; \mathrm{Lip}(C[0,\infty); \alpha, 1); x)$, $0 < \alpha \leqslant 1$	$n^{-\alpha/2}$	$\dfrac{\Gamma((\alpha+1)/2)}{\sqrt{\pi}}(2x(1-cx))^{\alpha/2}$
$\Delta(T_n; \mathrm{Lip}^*(C[0,\infty); \alpha; 2); x)$, $0 < \alpha \leqslant 2$	?	?
$\Delta(T_n; W^{(1)}(C[0,\infty); \alpha; 1); x)$, $0 < \alpha < 1$	$n^{-(\alpha+1)/2}$	?
$\Delta(T_n; W^{(1)}(C[0,\infty); 1; 1); x)$,	n^{-1}	$\dfrac{1}{2}x(1-cx)$
$\Delta(T_n; \alpha; q; x)$, q even, $0 < \alpha \leqslant 1$	$n^{-(\alpha+q)/2}$	$\dfrac{\Gamma((\alpha+q+1)/2)}{(1+\alpha)_q\sqrt{\pi}}(2x(1-cx))^{(\alpha+q)/2}$
$\Delta(T_n; \alpha; q; x)$, q odd, $0 < \alpha < 1$	$n^{-(\alpha+q)/2}$	?
$\Delta(T_n; 1; q; x)$, q odd	$n^{-(q+1)/2}$	$\dfrac{\Gamma((q+2)/2)}{(q+1)!\sqrt{\pi}}(2x(1-cx))^{(q+1)/2}$

(b) for the sequence $(\tilde{T}_n)_{n \in \mathbb{N}}$

	Order of approximation	Local Nikolskii constant
$\Delta(\tilde{T}_n : \mathrm{Lip}(C(I); \alpha; 1); x)$, $0 < \alpha \leqslant 1$	$n^{-\alpha/2}$	$\frac{\Gamma((\alpha+1)/2)}{\sqrt{\pi}} (2x(1-cx))^{\alpha/2}$
$\Delta(\tilde{T}_n : \mathrm{Lip}^*(C(I); \alpha; 2); x)$, $0 < \alpha \leqslant 2$	$n^{-\alpha/2}$	$\frac{\Gamma((\alpha+1)/2)}{\sqrt{\pi}} (2x(1-cx))^{\alpha/2}$
$\Delta(\tilde{T}_n : W^{(1)}(C(I); \alpha; 1); x)$, $0 < \alpha \leqslant 1$	$n^{-(\alpha+1)/2}$	$\frac{2^{\alpha-1}\Gamma((\alpha+2)/2)}{\sqrt{\pi}} (2x(1-cx))^{(\alpha+1)/2}$
$\Delta(\tilde{T}_n ; \alpha; q; x)$, q even, $0 < \alpha \leqslant 1$	$n^{-(\alpha+q)/2}$	$\frac{\Gamma((\alpha+q+1)/2)}{(1+\alpha)_q \sqrt{\pi}} (2x(1-cx))^{(\alpha+q)/2}$
$\Delta(\tilde{T}_n ; \alpha; q; x)$, q odd, $0 < \alpha \leqslant 1$	$n^{-(\alpha+q)/2}$	$\frac{2^{\alpha-1}\Gamma((\alpha+q+1)/2)}{(1+\alpha)_q \sqrt{\pi}} (2x(1-cx))^{(\alpha+q)/2}$

where $(1+\alpha)_q := \prod_{k=1}^{q} (\alpha + k)$.

Remark. From Theorem 2.2 we have learned that the above results are valid

(a) for $0 < x < \min(b, 1/(2c))$, if the sequence $(\Phi_n)_{n \in \mathbb{N}}$ satisfies all conditions of Definition 1.1 on an interval $[0, b]$ with $c > 0$, (in particular for $0 < x < 1$, if $c = 1$, $\Phi_n(x) = (1-x)^n$);

(b) for all $x > 0$, if $\Phi_n(x) = \exp(-nx)$ $(c = 0)$ or $\Phi_n(x) = (1 - cx)^{n/c}$ $(c < 0)$.

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